Automated Verification of Cyber-Physical Systems

A.A. 2022/2023

Corso di Laurea Magistrale in Informatica

Statistical Model Checking

Igor Melatti

Università degli Studi dell'Aquila

Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Statistical Model Checking

- In the recent literature, there is not full consensus, as it may refer to:
    - estimate the probability of a BLTL property of a stochastic system
        - in a stochastic system, transitions between states are not deterministic, but probabilistic
    - estimate the probability of an LTL property of a deterministic system
        - thus, input is the same of a classical model checking problem
        - but output is "probabilistic"
        - also referred to as *quantitative model checking* or *monte-carlo model checking*

Some Background

- In our context, a *random variable* is a function from some event space Ω to $\mathbb{R}$
  - $X : \Omega \to \mathbb{R}$
- Suppose we have a probability $\mathbb{P}$ defined on 2^Ω
  - thus, $\mathbb{P}$ is defined on sets of events $E \subseteq \Omega$
  - recall the Kolmogorov axioms:
    - $\forall E \subseteq \Omega.\ \mathbb{P}(E) \in [0, 1]$
    - $\mathbb{P}(\Omega) = 1$
    - $\forall I \subseteq \mathbb{N} : (E_i \in 2^\Omega \wedge\ \forall i \neq j \in I.\ E_i \cap E_j = \varnothing) \ \to\ \mathbb{P}(\cup_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$
- The *mean* of a random variable, also called *expected value*, is defined as $\mu_X = \mathbb{E}[X] = \sum_{\omega \in \Omega} p(\omega) X(\omega)$
  - here $p(\omega) = \mathbb{P}(\{\omega\})$
  - by the axioms above, $p(\omega) \in [0, 1]$ and $\sum_{\omega \in \Omega} p(\omega) = 1$
  - for $|\Omega| < \infty$, this is a weighted average...
- Example: ...

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Some Background

- Suppose that $|\Omega| = 2$, i.e., we have just two possible outcomes
  - without loss of generality, $\Omega = \{0, 1\}$
  - again, $p(0)$ and $p(1)$ are defined in some way
  - for sure, $p(1) = 1 - p(0)$; often $p(1)$ is simply p and $p(0)$ is $q = 1 - p$
- A *Bernoulli random variable* Z on Ω is s.t. $Z : \{0, 1\} \to \{0, 1\}$, $Z(x) = x$
  - we simply write Z instead of $Z(x)$
  - given $\mathbb{P}$ on 2^Ω, we define $p_Z = \mathbb{P}(Z = 1)$
  - following the notation above, $\mu_Z = \mathbb{E}[Z] = Z(1)p_Z + Z(0)q_Z = p_Z$

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Some Background

- A *Bernoulli process* consists in repeatedly running independent trials on a Bernoulli variable Z
  - either finite or infinite sequence of trials
  - "independent" means that the probability of outcome $o_1 \dots o_n$ is $\prod_{i=1}^{n} p(o_i)$
  - if there are k outcomes such that $o_i = 1$, then $\mathbb{P}(\{o_1 \dots o_n\}) = p_Z^k q_Z^{n-k} = p_Z^k (1 - p_Z)^{n-k}$
- We can define a geometric random variable X_Z s.t. $X : \Omega^\infty \to \mathbb{N}$
  - $X_Z(\omega) = n$ iff $Z = 1$ for the first time after exactly n independent trials (with probability p_Z)
- Thus, $\mathbb{P}(X_Z = N) = q_Z^{N-1} p_Z$
  - as a consequence, $\mathbb{P}(X_Z \leq N) = \sum_{n \leq N} q_Z^{N-1} p_Z = 1 - q_Z^N$

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

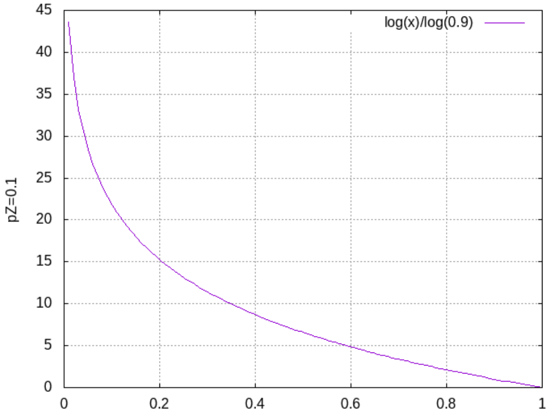
DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Some Background

- Suppose now that, in some way, you know the value of p_Z
- How many trials would we need to see $Z = 1$?
- Well, in these terms, you would need infinitely many trials
  - special case 1: you can't see $Z = 1$ if $p_Z = 0$
  - special case 2: you see $Z = 1$ after 1 trial if $p_Z = 1$
  - we are interested in $0 < p_Z < 1$
- Let's relax a bit: how many trials would we need to see $Z = 1$ *with a given confidence* $1 - \delta$?
  - e.g.: I want to be 90% sure, so $\delta = 0.1$
- We have $\mathbb{P}(X_Z \leq N) = 1 - (1 - p_Z)^N \geq 1 - \delta$
  - solving N as a function of δ and p_Z, we have $N \geq \frac{\log(\delta)}{\log(1-p_Z)}$
  - note that both numerator and denominator are negative, as $0 < \delta, p_Z < 1$

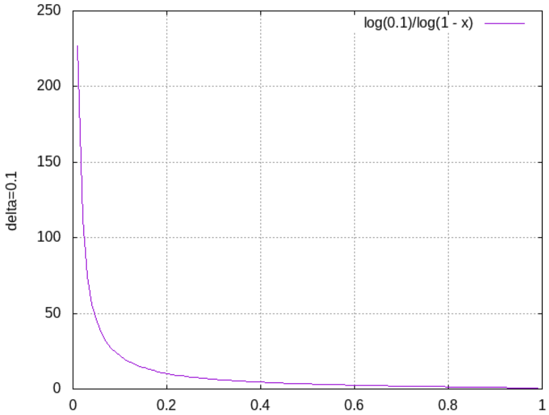
Some Background

- For fixed p_Z, N decreases with δ
  - i.e., increases with $1 - \delta$
  - you are ok if you are less confident? you can try less
  - you want to be more confident? you have to try more



Some Background

- For fixed δ, N decreases with p_Z
  - you want to detect something with big probability? you can try less
  - you want to detect something with small probability? you have to try more



Some Background

- ... But we do not know p_Z
  - indeed, it is exactly what we want to estimate by making trials
- Again, getting at the precise value p_Z is too difficult, but we can choose an accuracy
  - we may choose some value $\varepsilon > 0$ of interest and test if $p_Z \geq \varepsilon$
  - ε is our *error margin*, $H_0 \equiv (p_Z \geq \varepsilon)$ is the *null hypothesis*
- We have that $M = \frac{\log(\delta)}{\log(1-\varepsilon)} \geq \frac{\log(\delta)}{\log(1-p_Z)} = N$
- Recalling the steps before, we have $\mathbb{P}(X_Z \leq M) \geq \mathbb{P}(X_Z \leq N) \geq 1 - \delta$
- Thus: $p_Z \geq \varepsilon$ *implies* $\mathbb{P}\left(X_Z \leq \frac{\log(\delta)}{\log(1-\varepsilon)}\right) \geq 1 - \delta$
  - using conditional probabilities and putting M back in, we have $\mathbb{P}\left(X_Z \leq M \mid p_Z \geq \varepsilon\right) \geq 1 - \delta$

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Some Background

- Suppose we want to decide if $H_0 \equiv p_Z \geq \varepsilon$ holds (*hypothesis testing*)
- We perform $M = \left\lceil \frac{\log(\delta)}{\log(1-\varepsilon)} \right\rceil$ trials on Z
  - if we never see $Z = 1$, then we reject H_0
  - otherwise, we accept H_0
- There are 4 possible "higher outcomes"
  - *type-I error*: H_0 is rejected, but $p_Z \geq \varepsilon$ holds
  - *type-II error*: H_0 is accepted, but $p_Z < \varepsilon$ holds
  - we were right in rejecting/accepting H_0 (2 cases)
- The probability of a type-I error is denoted by α, the probability of a type-II error is β
  - generally speaking, they could be dependent on each other
- We have $\alpha = \mathbb{P}(X > M \mid H_0) = 1 - \mathbb{P}(X \leq M \mid H_0) \leq \delta$
  - since $\mathbb{P}(X \leq M \mid H_0) \geq 1 - \delta$

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

LTL Monte-Carlo Model Checking

- Grosu, Smolka: "Monte Carlo Model Checking", Proc. of TACAS 2005
- In LTL Monte-Carlo Model Checking, the first part of the input is as in standard LTL Model Checking:
  - a Kripke structure $\mathcal{S} = \langle S, I, R, L \rangle$
  - an LTL formula φ
  - let us say that we directly have the Büchi Automaton $B = B_{\neg\varphi} \times B_{\mathcal{S}}$
    - as it is computed by explicit on-the-fly model checkers like SPIN
- Then, we also have two additional inputs: $0 < \delta, \varepsilon < 1$
- Output as in standard LTL Model Checking:
  - either PASS...
  - ... or FAIL with a counterexample

LTL Monte-Carlo Model Checking

- If FAIL with a counterexample σ is returned, then for sure we have an error in our model
  - that is, $\mathcal{S} \not\models \varphi$ holds
  - σ is a counterexample showing that $\mathcal{S} \not\models \varphi$
- Otherwise, it may still be the case that, notwithstanding the PASS result, $\mathcal{S} \not\models \varphi$
- However, the probability that $\mathcal{S} \not\models \varphi$ is less than δ
  - indeed, this does only work with a huge assumption (which involves the remaining input ε), as we will see
  - however, the huge assumption could be made reasonable
- How is this achieved? Exactly through the steps outlined above!

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

LTL Monte-Carlo Model Checking

- Recall that a (non-deterministic) Büchi Automaton (BA) is a 5-tuple $B = \langle \Sigma, Q, \delta, Q_0, F \rangle$ where:
  - Σ is the *alphabet*, i.e., a finite set of symbols
  - Q is the finite set of states, $Q_0 \subseteq Q$ are the initial states and $F \subseteq Q$ are the final states
  - $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation
- We suppose that $B = B_{\neg\varphi} \times B_{\mathcal{S}}$ is the Cartesian product of the Kripke structure $\mathcal{S}$ and the Büchi automaton generated from φ using known algorithms
  - e.g., as it is implemented in SPIN
- A *lasso* of B is a sequence $\sigma = q_0 x_0 q_1 \ldots q_n$ s.t.:
  - $\forall 0 \leq i < n.\ (q_i, x_i, q_{i+1}) \in \delta$
  - $\exists 0 \leq k \leq n :\ \forall 0 \leq i, j < n-1.\ q_i \neq q_j \wedge q_n = q_k$
- A lasso is *accepting* if $\exists k \leq i \leq n :\ \ q_i \in F$

LTL Monte-Carlo Model Checking

- We may easily define a probability distribution on the finite runs σ of B:
  - $\mathbb{P}(q_0) = \frac{1}{|Q_0|}$
  - $\mathbb{P}(q_0 x_0 q_1 \dots q_{n-1} x_{n-1} q_n) = \mathbb{P}(q_0 x_0 q_1 \dots q_{n-1}) \frac{1}{|\delta(q_{n-1})|}$
  - being $\delta(q) = \{(q, x, q') \mid (q, x, q') \in \delta\}$
  - that is: each time we have a (non-deterministic) choice, we choose one uniformly at random
- Such probability is well-defined: we may extend it to obtain a (discrete) probability space $(2^L, \mathbb{P})$
  - being $L = \{\sigma \mid \sigma \text{ is a lasso in } B\}$
  - furthermore, $L \supseteq L_a = \{\sigma \mid \sigma \text{ is an accepting lasso in } B\}$
  - $L_n = L \setminus L_a$ is the set of non-accepting lassos

LTL Monte-Carlo Model Checking

- Given $(2^L, \mathbb{P})$, our Bernoulli variable Z is defined by:
  - take one lasso σ from L, following the rules defined by $\mathbb{P}$
  - that is: make a random walk (see the algorithm below)
  - $Z = 1$ iff $\sigma \in L_a$ is accepting
- From a theoretical point of view, since $|L| < \infty$, we would be tempted to say that $p_Z = \frac{|L_a|}{|L|}$
- But this is not true, since lassos do not have the same probability, according to $\mathbb{P}$
- Thus, $p_Z = \mathbb{P}(Z = 1) = \sum_{\lambda_a \in L_a} \mathbb{P}(\lambda_a)$
  - not actually useful for computation: L_a requires generating L, which may run out of computational resources

LTL Monte-Carlo Model Checking

```
$MC^2$(KS_BA $\mathcal{SA}$, double $\varepsilon$, double $\delta$) {
  for i in 1..$\left\lceil \frac{\log(\delta)}{\log(1-\varepsilon)} \right\rceil$
    if (SampleLasso($\mathcal{SA}$) == (1, $\sigma$))
      return (FAIL, $\sigma$);
  return PASS;
}
SampleLasso(KS_BA $\mathcal{SA} = \langle \Sigma, Q, \delta, Q_0, F \rangle$) {
  $(i, f, H, q)$ = $(0, 0, \varnothing,$ pick_unif_random($Q_0$);
  while($H(q) = \perp$) {
    $H(q)$ = i + 1; i = i + 1;
    if ($q \in F$) f = i;
    $q$ = pick_unif_random($\delta(q)$);
  }
  if ($H(q) \leq$ f) return (1, getCurrLasso($H$));
  else return (0, $\perp$);
}
```

Nested DFS for LTL Model Checking

```
DFS(KS_BA $\mathcal{SA}$, state $(s,q)$, bool n, state a) {
  let $\mathcal{SA} = \langle S_A, I_A, R_A, L_A \rangle$;
  foreach $(s', q') \in S_A$ s.t. $((s,q),(s',q')) \in R_A$ {
    if (n $\wedge$ $(s,q)$ == a)
      exit reporting error;
    if ($(s', q'$,n$) \notin T$) {
      $T$ = $T \cup \{(s', q'$,n$)\}$;
      DFS($\mathcal{SA}$, $(s', q')$, n, a);
      if ($\neg$n $\wedge$ $(s', q')$ is accepting) {
        DFS($\mathcal{SA}$, $(s', q')$, true, $(s', q')$);
} } } }

LTLMC(KS $\mathcal{S}$, LTL $\varphi$) {
  $\mathcal{A}$ = BA_from_LTL($\varphi$); $T$ = $\varnothing$;
  let $\mathcal{S} = \langle S, I, R, L \rangle$, $\mathcal{A} = \langle \Sigma, Q, \delta, Q_0, F \rangle$;
  foreach $s \in I, q \in Q_0$
    DFS($\mathcal{S} \times \mathcal{A}$, $(s,q)$, false, null);
}
```

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

LTL Monte-Carlo Model Checking

- Standard LTL Model Checking requires both time and space to be at least $O(|S|)$
  - easily billion of states, often unaffordable for real-world systems
- Here, time is $O(MD)$ and space is $O(D)$
  - being D the diameter of S, i.e., the length of the longest lasso starting from an initial state
  - $M = \left\lceil \frac{\log(\delta)}{\log(1-\varepsilon)} \right\rceil$ as usual
- No type-II errors: if we find a counterexample, we are happy
- Given the discussion on the background, if the answer is PASS, then the probability that an error is present but came undetected through the M trials is less than δ
- However, this is only true if we *assume* that $p_Z \geq \varepsilon$

LTL Monte-Carlo Model Checking: $p_Z \geq \varepsilon$

- Recall that $Z = 1$ iff, making a random walk on the given BA, I find an accepting lasso
  - recall also that an accepting lasso is “bad”, i.e., the property does not hold in the system
- Thus, we are saying that the probability that, among all lassos I can find with a random walk, the probability that it is accepting is at least ε
- There are two cases:
  - $p_Z < \varepsilon$, e.g., because there are not errors in the starting system, thus $p_Z = 0$
  - well, if this is true, the probability of that a counterexample exists is less than ε
  - $p_Z \geq \varepsilon$, then the probability that a counterexample exists is less than δ by the discussion above
  - note however that such two probabilities are not defined in the same way

LTL Monte-Carlo Model Checking: Experimental Results

- Results for classical systems: dining philosophers and Needham-Schroeder protocol
  - for dining philosophers, two properties: one is false (with counterexample), one is false
  - Needham-Schroeder is the bugged version (with counterexample)
  - $\delta = 0.1, \varepsilon = 0.0018 \rightarrow M = 1257$
- Columns meaning:
  - `ph`: number of philosophers
  - `mr`: parameter in the Needham-Schroeder protocol
    - the bigger the value, the bigger the number of states
  - `entr`: number of entries in the hash table (RAM usage...)
  - `mxl`: max length of a lasso
  - `cxl`: length of the counterexample found
  - `M`: number of trials to find a counterexample

Unfair Dining Philosophers

	DDFS		MC^2			
ph	time	entr	time	mxl	cxl	M
4	0.02	31	0.08	10	10	3
8	1.62	511	0.20	25	8	7
12	3:13	8191	0.25	37	11	11
16	>20:0:0	–	0.57	55	8	18
20	–	oom	3.16	484	9	20
30	–	oom	35.4	1478	11	100
40	–	oom	11:06	13486	10	209

	DDFS		MC^2			
ph	time	entr	time	mxl	cxl	M
4	0.17	29	0.02	8	8	2
8	0.71	77	0.01	7	7	1
12	1:08	125	0.02	9	9	1
16	7:47:0	173	0.11	18	18	1
20	–	oom	0.06	14	14	1
30	–	oom	1.12	223	223	1
40	–	oom	1.23	218	218	1

Fair Dining Philosophers

	DDFS		MC^2		
ph	time	entr	time	mxl	avl
4	0:01	178	0:20	49	21
6	0:03	1772	0:45	116	42
8	0:58	18244	2:42	365	99
10	16:44	192476	7:20	720	234
12	–	oom	21:20	1665	564
16	–	oom	3:03:40	7358	3144
20	–	oom	19:02:00	34158	14923

	DDFS		MC^2		
ph	time	entr	time	mxl	avl
4	0:01	538	0:20	50	21
6	0:17	9106	0:46	123	42
8	7:56	161764	2:17	276	97
10	–	oom	7:37	760	240
12	–	oom	21:34	1682	570
16	–	oom	2:50:50	6124	2983
20	–	oom	22:59:10	44559	17949

Unfair Needham-Schroeder

	DDFS		MC^2			
mr	time	entr	time	mxl	cxl	M
4	0.38	607	1.68	87	87	103
8	1.24	2527	11.3	208	65	697
16	5.87	13471	10.2	223	61	612
24	18.7	39007	3:06	280	44	12370
32	36.2	85279	2:54	269	63	11012

	DDFS		MC^2			
mr	time	entr	time	mxl	cxl	M
40	1:11	158431	1:46	325	117	7818
48	2:03	264607	1:45	232	25	6997
56	3:24	409951	6:54	278	133	28644
64	5:18	600607	7:12	347	32	29982
72	–	oom	11:53	336	63	43192

LTL Quantitative Model Checking

- Grosu, Smolka: "Quantitative Model Checking", Proc. of ISOLA 2004
- Input is the same as before: a KS $\mathcal{S}$, an LTL formula φ, $0 < \delta, \varepsilon < 1$
  - again, let's say we have $B = B_{\neg\varphi} \times B_{\mathcal{S}}$
- Output is the same: PASS or (FAIL, counterexample)
- FAIL is FAIL as before
- Much easier interpretation for PASS: as we will see, with confidence $1 - \delta$ we have a bound ε on the probability of $\mathcal{S} \not\models \varphi$

LTL Quantitative Model Checking

- Let Z be a random variable with values in $[0, 1]$
  - thus, Z is generally *not* a Bernoulli variable
  - but Bernoulli variables are a special case, so the methodology discussed below can be applied
- Recall that the *mean* of Z is $\mu_Z = \mathbb{E}[Z] = \sum_{\omega \in \Omega} p(\omega)Z(\omega) \in [0, 1]$
  - recall that, if Z is a Bernuolli variable, $\mu_Z = p_Z$
- The purpose here is exactly to compute μ_Z
- The exact value cannot be directly computed, so let us say we output $\tilde{\mu}_Z$ instead
- The methodology proposed here ensures that $\mathbb{P}(\mu_Z(1 - \varepsilon) \leq \tilde{\mu}_Z \leq \mu_Z(1 + \varepsilon)) \geq 1 - \delta$
  - so again, ε is a tolerance and δ is a confidence on the result
  - typically, they should be close to zero
  - often, this is called a (ε, δ)-approximation

OAA: Optimal Approximation Algorithm

- Dagum, Karp, Luby, Ross: "An Optimal Algorithm for Monte Carlo Estimation". *SIAM Journal on Computing, 29(5):1484–1496*, 2000.
- We have Z as a random variable in $[0, 1]$: how do we compute an (ε, δ)-approximation $\tilde{\mu}_Z$ of μ_Z?
- Idea: perform N independent trials of Z, callect results $Z_1, \ldots, Z_n$ and then output $\tilde{\mu}_Z = \frac{\sum_{i=1}^{N} Z_i}{N}$
- Straightforward problem: how to choose N, so as we have an (ε, δ)-approximation?

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

OAA: Optimal Approximation Algorithm

- We may employ an algorithm which *dynamically adjusts* the value of N on the basis of the results obtained so far
- In doing so, we use an auxiliary function SRA (Stopping Rule Algorithm)
- We also suppose to have a procedure $\mathcal{P}_Z$ which performs an experiment on Z and returns the corresponding value in $[0, 1]$
  - of course, different calls to $\mathcal{P}_Z$ will return different values
- Big limitation: $\mu_Z > 0$, or SRA does not terminate

OAA: Optimal Approximation Algorithm

```
OAA(procedure_for_Z $\mathcal{P}_Z$, double $\varepsilon$, double $\delta$) {
  $\hat{\mu}_Z$ = SRA($\mathcal{P}_Z, \min\{\frac{1}{2}, \sqrt{\varepsilon}\}, \frac{\delta}{3}$);
  $\Upsilon$ = $2(1+\sqrt{\varepsilon})(1+2\sqrt{\varepsilon})\left(1+\frac{\log(3)-\log(2)}{\log(2)-\log(\delta)}\right)\frac{4(e-2)(\log(2)-\log(\delta))}{\varepsilon^2}$;
  $N$ = $\frac{\varepsilon\Upsilon}{\hat{\mu}_Z}$;
  $S$ = $\frac{1}{2}\sum_{i=1}^{N}(\mathcal{P}_Z() - \mathcal{P}_Z())^2$;
  $\rho_Z$ = $\max\left\{\frac{S}{N}, \varepsilon\hat{\mu}_Z\right\}$;
  $N$ = $\frac{\rho_Z\Upsilon}{\hat{\mu}_Z^2}$;
  $S$ = $\frac{1}{N}\sum_{i=1}^{N}\mathcal{P}_Z()$;
  return $\tilde{\mu}_Z = \frac{S}{N}$;
}
```

SRA: Stopping Rule Algorithm

```
SRA(procedure_for_Z $\mathcal{P}_Z$, double $\varepsilon$, double $\delta$) {
    $\Upsilon$ = $1 + (1+\varepsilon)\frac{4(e-2)(\log(2)-\log(\delta))}{\varepsilon^2}$;
    $N$ = 1;
    $S$ = 0;
    while ($S$ $\leq$ $\Upsilon$) {
        $N$ = $N$ + 1;
        $S$ = $S$ + $\mathcal{P}()$;
    }
    return $\hat{\mu}_Z = \frac{S}{N}$;
}
```

LTL Quantitative Model Checking

- Leveraging on OAA, we use the almost same framework used for Monte-Carlo Model Checking
  - Bernoulli variable Z s.t. $Z = 1$ iff, making a random walk, you detect an non-accepting lasso
  - note that we reversed the previous definition: we will be back on this
  - Z is a special case of the random variables of OAA, so we may apply OAA to Z
  - also the probability space $(2^L, \mathbb{P})$ is the same
- The subroutine `SampleLasso` is the same as above

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

LTL Quantitative Model Checking

- OAA* is a modified version of OAA: as soon as $Z = 0$ for some trial, exit with probability 0
  - i.e., if a counterexample has been found
- Thus, OAA* returns either 0 (in the previous case) or 1 (otherwise)

```
QMC(KS_BA $\mathcal{SA}$, **double** $\varepsilon$, **double** $\delta$) {
  $\tilde{p}_Z$ = OAA*(SampleLasso($\mathcal{SA}$), $\varepsilon$, $\delta$);
  **if** ($\tilde{p}_Z$ == 0) {
    $\sigma$ = extract the accepting lasso from the
          last trial;
    **return** (FAIL, $\sigma$);
  }
  **else**
    **return** PASS;
}
```

LTL Quantitative Model Checking

- Why $Z = 1$ if we find a “good” lasso?
  - instead of $Z = 1$ if we find a counterexample, as it was for MC^2?
- Recall that OAA only works if $p_Z = \mu_Z > 0$, otherwise SRA does not terminate
- With the current definition, $p_Z > 0$ means “there is at least a good lasso”
  - with the MC^2 definition, $p_Z > 0$ means “there is at least a counterexample”: could easily be false!
- Even if “there is at least a good lasso” is false, QMC terminates as OAA* immediately exit after the first trial...

LTL Quantitative Model Checking

- Recall that $\mathbb{P}(\mu_Z(1-\varepsilon) \leq \tilde{\mu}_Z \leq \mu_Z(1+\varepsilon)) \geq 1-\delta$
- If $Z = 1$ for all trials, $\tilde{\mu}_Z = 1$ -
- Thus, $1 = \tilde{\mu}_Z \geq \mu_Z(1-\varepsilon)$ always holds
- What remains is $\mathbb{P}(\mu_Z(1+\varepsilon) \geq 1) \geq 1-\delta$
  - better: $\mathbb{P}(\mu_Z \geq \frac{1}{1+\varepsilon}) \geq 1-\delta$
- If we recall that $\mu_Z = p_Z = 1 - q_Z$ we have that $\mathbb{P}(q_Z \leq 1 - \frac{1}{1+\varepsilon}) = \mathbb{P}(q_Z \leq \frac{\varepsilon}{1+\varepsilon}) \geq 1-\delta$
- Actually, for small ε, $\frac{\varepsilon}{1+\varepsilon} \approx \varepsilon$, thus we are saying that $\mathbb{P}(q_Z \leq \varepsilon) \geq 1-\delta$!
  - q_Z is the probability that, making a random walk, we find a counterexample
  - much better than the obscure assumption of MC^2

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

LTL Quantitative Model Checking

- QMC seems extremely better than MC^2
- So why MC^2 has been published as an improvement of QMC one year later?
- Because the OAA methodology requires much more steps
- For MC^2, the worst-case number of trials is $M = \frac{\log(\delta)}{\log(1-\varepsilon)}$
- For QMC, we can show that worst-case number of trials is bound by $N = O(4\frac{\log(2)-\log(\delta)}{\varepsilon})$
    - recall that $\log(\delta) < 0$
    - $N > 5M$
    - e.g., $\delta = 0.1, \varepsilon = 0.0018 \rightarrow M = 1257$ for MC^2
    - but $N = 1257$ with $\delta = \varepsilon = 0.1$ for QMC
- RAM space is $O(D)$ for both

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Unfair Dining Philosophers

	DDFS		QMC			
ph	time	entr	time	mxl	cxl	N
4	0.02	31	0.08	10	10	3
8	1.62	511	0.20	25	8	7
12	3:13	8191	0.25	37	11	11
16	>20:0:0	–	0.57	55	8	18
20	–	oom	3.16	484	9	20
30	–	oom	35.4	1478	11	100
40	–	oom	11:06	13486	10	209

	DDFS		QMC			
ph	time	entr	time	mxl	cxl	N
4	0.17	29	0.02	8	8	2
8	0.71	77	0.01	7	7	1
12	1:08	125	0.02	9	9	1
16	7:47:0	173	0.11	18	18	1
20	–	oom	0.06	14	14	1
30	–	oom	1.12	223	223	1
40	–	oom	1.23	218	218	1

Fair Dining Philosophers

	DDFS		QMC		
phi	time	entries	time	mxl	avl
4	0:01	178	0:20	49	21
6	0:03	1772	0:45	116	42
8	0:58	18244	2:42	365	99
10	16:44	192476	7:20	720	234
12	–	oom	21:20	1665	564
14	–	oom	1:09:52	2994	1442
16	–	oom	3:03:40	7358	3144
18	–	oom	6:41:30	13426	5896
20	–	oom	19:02:00	34158	14923

	DDFS		QMC		
phi	time	entries	time	mxl	avl
4	0:01	538	0:20	50	21
6	0:17	9106	0:46	123	42
8	7:56	161764	2:17	276	97
10	–	oom	7:37	760	240
12	–	oom	21:34	1682	570
14	–	oom	1:09:45	3001	1363
16	–	oom	2:50:50	6124	2983
18	–	oom	8:24:10	17962	7390
20	–	oom	22:59:10	44559	17949

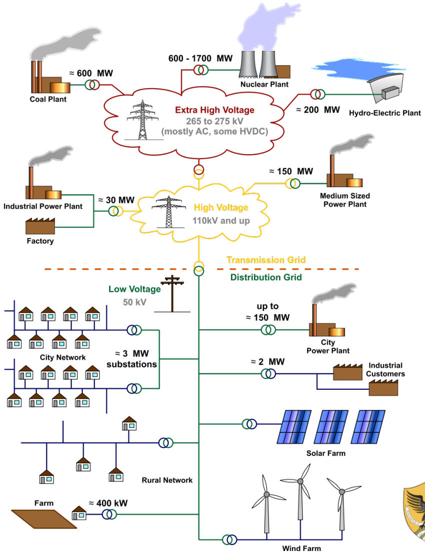
Combining QMC and MC^2: a Case Study

- Mancini, Mari, Melatti, Salvo, Tronci, Gruber, Hayes, Prodanovic, and Elmegaard. "Parallel Statistical Model Checking for Safety Verification in Smart Grids." In Proc. SmartGridComm 2018.
- EDN: Electric Distribution Network, also called "grid"
  - brings to residential houses, commercial buildings and industries the electricity they need
  - till some decades ago, simply based on demands
- Smart grid: usage of computational services to improve electricity distribution
  - e.g.: electricity usage is measured and then rendered in a web app

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Combining QMC and MC^2: a Case Study



UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

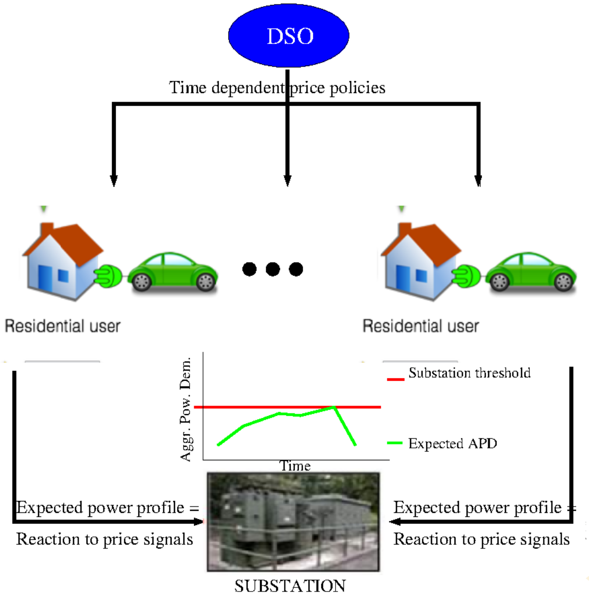
Combining QMC and MC^2: a Case Study

- Distribution System Operators (DSOs) and energy retailers compute price tariffs for residential users
- Expected Power Profiles (EPPs): how residential users will respond to price tariffs
- DSOs compute price tariffs so that EPPs do not threat substations safety
  - in each t, Aggregated Power Demand (APD) must be below the substation safety power threshold (e.g., 400 kW)
  - DSOs main goal is to achieve *peak shaving*

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Problem at a Glance



Autonomous Demand Response

- Residential users may or may not follow their corresponding Expected Power Profiles (EPPs)
  - there may be automatic tools to enforce EPPs
  - implemented on small devices on users premises
  - still, there is no guarantee, due to unexpected needs, bad forecasts, limited computational resources, etc.

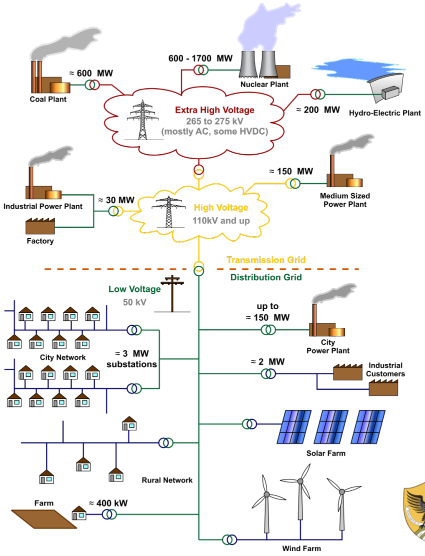
Problem

Given that users may deviate from EPPs with a given probability distribution, what is the resulting probability distribution for the aggregated power demand (APD)?

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

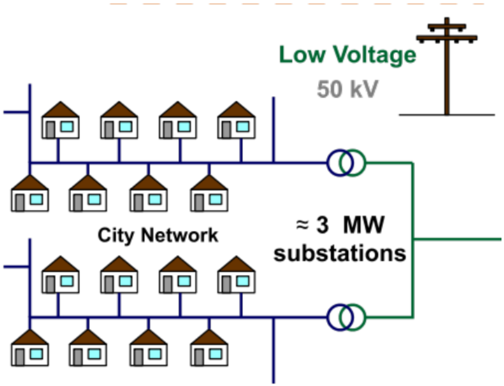
Combining QMC and MC^2: a Case Study



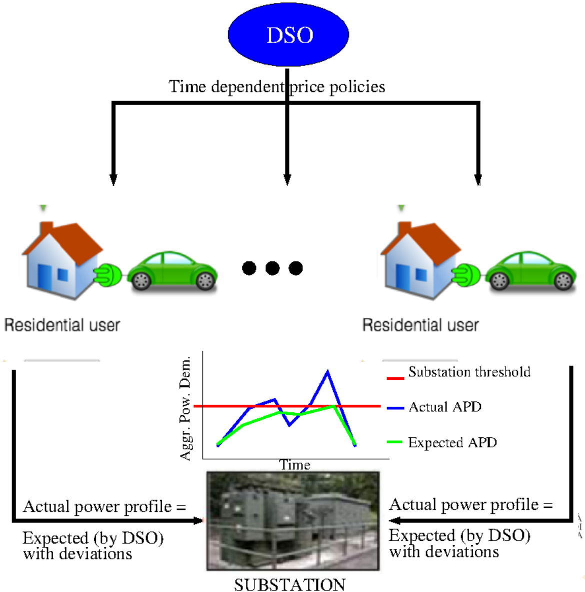
UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Combining QMC and MC^2: a Case Study



Problem at a Glance



APD-Analyser

- We present the APD-Analyser tool
  - APD: Aggregated Power Demand

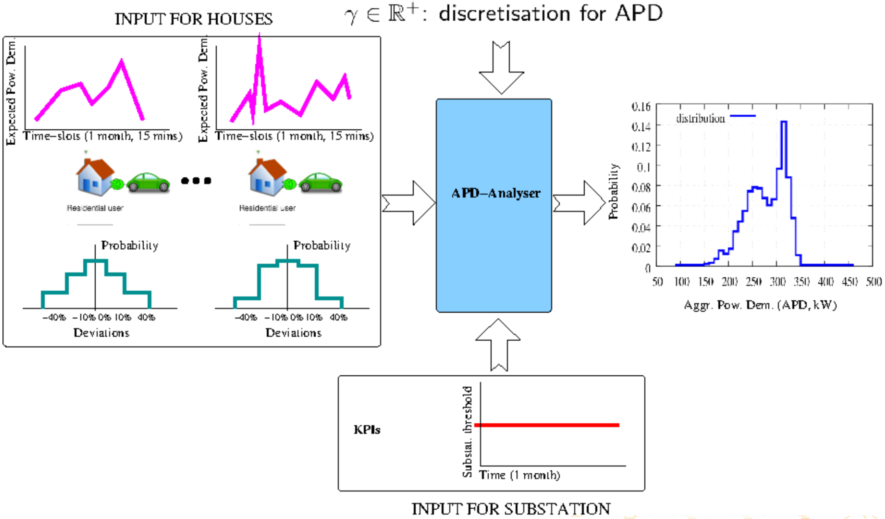
- Main goal: compute the probability distribution for the APD
  - given probability distributions on each residential user Expected Power Profile (EPP)

APD-Analyser: Input and Output

$0 < \varepsilon < 1$: tolerance

$0 < \delta < 1$: numerical accuracy

$\gamma \in \mathbb{R}^+$: discretisation for APD



APD-Analyser: Input

- Set of residential users U connected to the same substation
- Set of time-slots T (e.g., one month with 15 minutes step)
- Expected Power Profiles (EPP)
    - one for each user $u \in U$: for each time-slot $t \in T$, the expected power demand of u in t
    - $p_u : T \to \mathbb{R}$
- A probabilistic model for users deviations from EPPs
    - a real function $dev_u : \mathbb{R} \to [0, 1]$, for each user $u \in U$
    - $\int_{-\infty}^{+\infty} dev_u(x)dx = 1$
    - $\int_{a}^{b} dev_u(x)dx =$ probability that actual power demand of u in any time-slot $t \in T$ is in $[(1 + a)p_u(t), (1 + b)p_u(t)]$
    - e.g.: $\int_{-0.02}^{0.02} dev_u(x)dx =$ probability that actual power demand of u in any time-slot $t \in T$ deviates at most by 2% from EPP of u

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

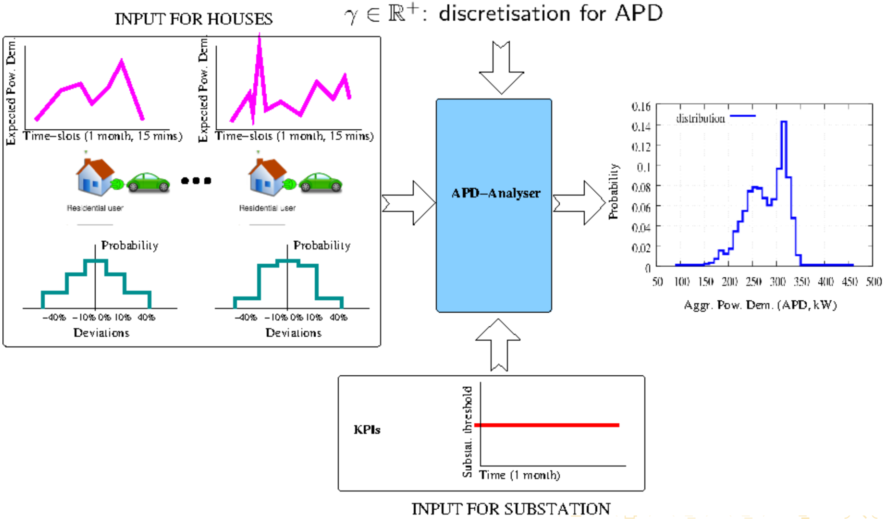
DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

APD-Analyser: Input and Output

$0 < \varepsilon < 1$: tolerance

$0 < \delta < 1$: numerical accuracy

$\gamma \in \mathbb{R}^+$: discretisation for APD



APD-Analyser: Input

- Substation safety requirements
  - $p_s : T \to \mathbb{R}$
  - for each $t \in T$, DSO wants the APD to be below $p_s(t)$
  - that is, $\forall t \in T \to \sum_{u \in U} [(1 + \mathrm{deviation}_u) p_u(t)] \leq p_s(t)$
- Key Performance Indicators (KPIs)
  - e.g., probability distribution that $p_s(t)$ is exceeded in any $t \in T$
- Parameters
  - $0 < \delta, \varepsilon < 1$: as for output probability distributions, the values must be correct up to tolerance ε with statistical confidence δ
    - $\Pr[(1 - \varepsilon)\mu \leq \tilde{\mu} \leq (1 + \varepsilon)\mu] \geq 1 - \delta$
    - μ: (unknown) correct value, $\tilde{\mu}$: computed value
  - $\gamma \in \mathbb{R}^+$: discretisation step for output probability distribution

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

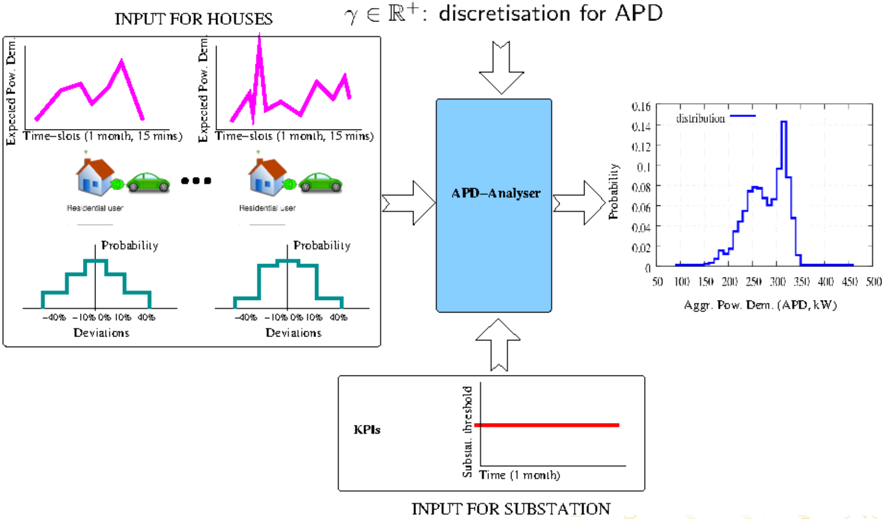
DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

APD-Analyser: Input and Output

$0 < \varepsilon < 1$: tolerance

$0 < \delta < 1$: numerical accuracy

$\gamma \in \mathbb{R}^+$: discretisation for APD



APD-Analyser: Output

- Probability distribution for APD resulting from EPPs disturbed with given probabilistic disturbance model
  - easy to evaluate KPIs once such distribution is computed
  - formally: $\Psi(W)$ is the probability that APD takes a value in interval W in any time-slot $t \in T$
- Exactly computing Ψ is infeasible, thus we compute $\tilde{\Psi}$ as a (ε, δ) approximation of a γ-discretisation of the APD
- For each γ-discretised value $w = \mathrm{APD}_{min} + k\gamma$, we compute $\tilde{\Psi}(w)$ s.t., with confidence at least $1 - \delta$:
  - if $\tilde{\Psi}(w) = \perp \notin [0, 1]$ then $\Psi([w, w + \gamma)) < \varepsilon$
  - otherwise, $\tilde{\Psi}([w, w + \gamma))$ is within $(1 \pm \varepsilon)\Psi(w)$

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

APD-Analyser: Algorithm

- *Monte-Carlo model checking*
  - goal: estimate the mean of a 0/1 random variable Z_w
  - $Z_w = 1$ iff, taken at random a $t \in T$, the value of the APD is in $[w, w + \gamma)$, when EPPs are perturbed using deviations model dev_u
  - then, the mean is exactly our $\tilde{\Psi}(w)$
- Method: perform N independent experiments (samples) for Z_w, and then the mean of Z_w is $\frac{\sum_{i=1}^{N} \hat{Z}_i}{N} \in [0, 1]$
  - Optimal Approximation Algorithm (OAA) by Dagum & al. (2000) + Monte-Carlo Model Checking (MCMC) by Grosu & Smolka (2005)
  - *sequential analysis*: use outcomes of previous experiments to compute N
  - the value of N is automatically adjusted, at run-time, while performing the samples
  - so that the desired tolerance ε is achieved with desired accuracy δ

Optimal Approximation Algorithm (OAA)

1. Phase 1
   1. Perform $N_1 = f_1(\varepsilon, \delta)$ experiments $\hat{Z}_{1,1}, \ldots, \hat{Z}_{1,N_1}$
   2. Compute mean of successful experiments $\hat{\mu}_Z = \frac{1}{N_1} \sum_{i=1}^{N_1} \hat{Z}_{1,i}$
2. Phase 2
   1. Perform $2N_2 = 2f_2(\varepsilon, \delta, \hat{\mu}_Z)$ experiments $\hat{Z}_{2,0}, \ldots, \hat{Z}_{2,2N_2-1}$
   2. Compute $S = \frac{1}{N_2} \sum_{i=0}^{N_2-1} \frac{|\hat{Z}_{2,2i} - \hat{Z}_{2,2i+1}|}{2}$
3. Phase 3
   1. Perform $N_3 = f_3(\varepsilon, \delta, \hat{\mu}_Z, S, N_2)$ experiments $\hat{Z}_{3,1}, \ldots, \hat{Z}_{3,N_3}$
   2. Return mean of successful experiments $\tilde{\mu}_Z = \frac{1}{N_3} \sum_{i=1}^{N_3} \hat{Z}_{1,i}$
      - It holds that $\Pr[(1-\varepsilon)\mu_Z \leq \tilde{\mu}_Z \leq (1+\varepsilon)\mu_Z] \geq 1 - \delta$

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

OAA + Monte-Carlo Model Checking (MCMC)

- Correct phase 1 using statistical hypothesis testing
- If $\sum_{i=1}^{M} \hat{Z}_{1,i} = 0$ for $M = f_4(\varepsilon, \delta) = \left\lceil \frac{ln(\delta)}{ln(1-\varepsilon)} \right\rceil$, terminate the computation
- Return $\tilde{\mu}_Z = 0$
- It holds that $\Pr[\mu_Z < \varepsilon] \geq 1 - \delta$

APD-Analyser: HPC Algorithm

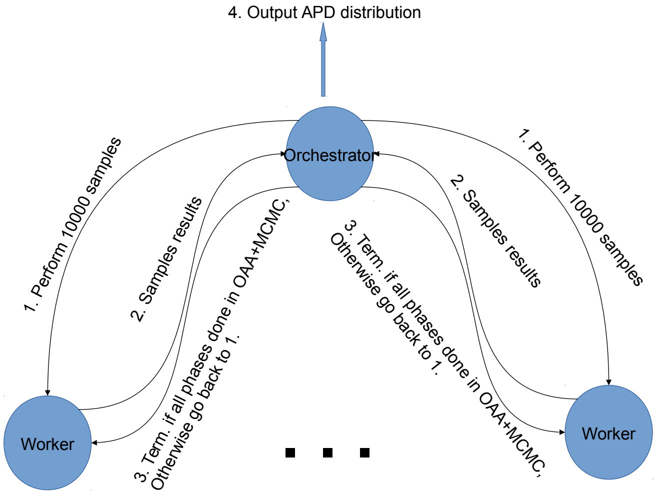
- $N = N_1 + N_2 + N_3$ can be prohibitively high
  - easily order of 10^9 in our experiments
  - OAA+MCMC to be run for each different value of w
  - if performed with a sequential algorithm, order of 1 month for the computation time
- We re-engineer the OAA to be run on a HPC infrastructure, i.e., a cluster (distributed memory)
  - main obstacle: value of N depends on samples outcomes! To be computed at run-time
- One *orchestrator* node instructs *worker* nodes to perform given number of samples
  - worker nodes perform samples in parallel and send results to the orchestrator
  - the orchestrator keeps track of phases of each worker and of different values of w
- As a result, less than 2 hours of computation with 89 workers

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

APD-Analyser: HPC Implementation Sketch

Different workers may be in different phases and different w


4. Output APD distribution
Orchestrator
1. Perform 10000 samples
2. Samples results
3. Term. if all phases done in OAA+MCMC, Otherwise go back to 1.
1. Perform 10000 samples
2. Samples results
3. Term. if all phases done in OAA+MCMC, Otherwise go back to 1.
Worker
Worker

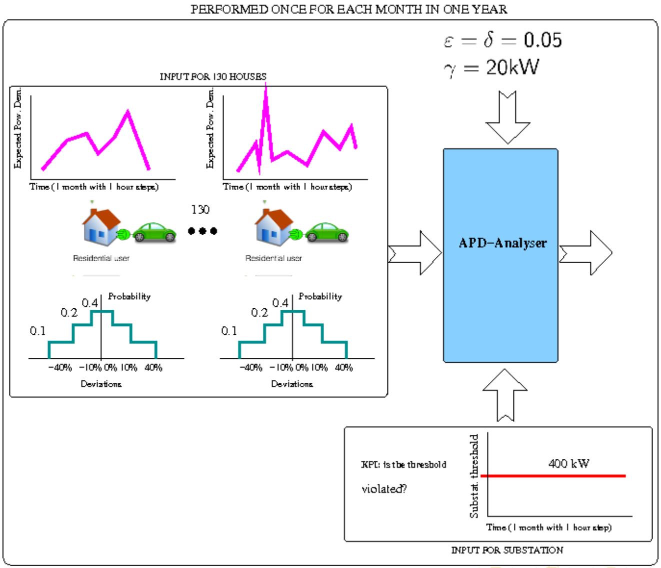
Experimental Evaluation: Case Study

- 130 houses in Denmark, all connected to the same substation
- EPPs computed by using methodologies from the literature
  - starting point: historical data collected on those houses for one year (SmartHG FP7 project)
  - computed as shifts within given flexibilities so as to collaboratively respond to price policies
- Very liberal deviation model: up to $\pm 40\%$ variations with 10% probability, up to $\pm 20\%$ variations with 20% probability
- We want to compute the APD for each month of the year
  - by using time-slots of 1 day, we have $5^{30 \times 130}$ overall number of deviations

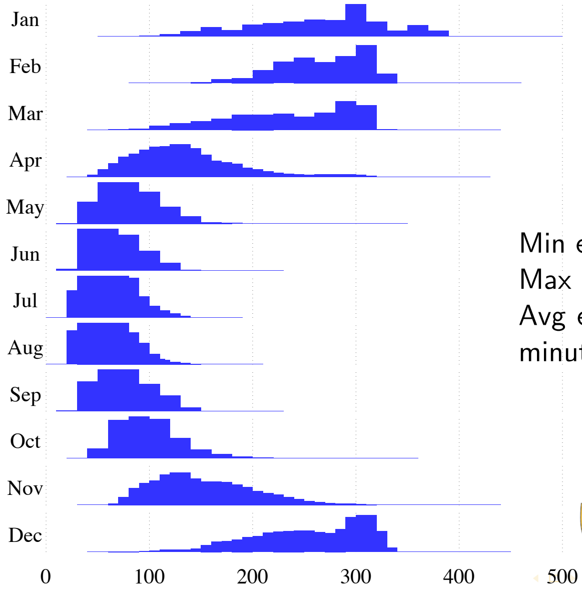
UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Experimental Evaluation: Case Study

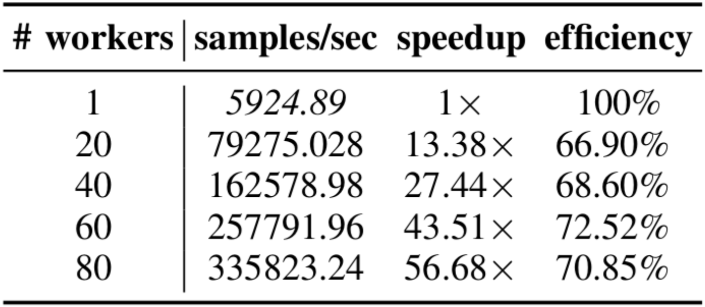


Experimental Results



Min exec time: 4782 secs
Max exec time: 6448 secs
Avg exec time: 1 hour, 28 minutes and 7 seconds

Experimental Results: HPC Scalability



# workers	samples/sec	speedup	efficiency
1	*5924.89*	1×	100%
20	79275.028	13.38×	66.90%
40	162578.98	27.44×	68.60%
60	257791.96	43.51×	72.52%
80	335823.24	56.68×	70.85%

$$\text{speedup} = \frac{s_k}{s_1}, \ \text{efficiency} = \frac{s_k}{k s_1}$$

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Conclusions

- We presented the HPC-based tool APD-Analyser
- Main purpose: support DSOs in analysing effects of price policies on aggregated power demand (APD) at substation level
    - especially for highly-fluctuating and individualised price policies
- From expected power profiles disturbed by probabilistic deviations (input) to probability distribution for APD (output)
- As a result, APD-Analyser enables safety assessment of price policies in highly dynamic ADR schemas

Statistical Model Checking for Everything

- Zuliani, Platzer, Clarke: "Bayesian Statistical Model Checking with Application to Stateflow/Simulink Verification", Formal Methods in System Design vol. 43, 2013
- In the works above, it was necessary to have some simple language defining the system
  - e.g., Promela of SPIN, though they use a different language
  - needed to perform the Cartesian product of the property and the system itself
  - and also to actually make a random walk of the system
  - actually, such a limitation is not difficult to overcome, but it is presented in this way
  - especially ok for systems already expressed in the language, but which went out of resources
- Here, we directly use simulators
  - Simulink, but conceptually also Modelica

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Some Background

- As before, we have to define our probability space; this time is not easy
- Given a set X, a *σ-algebra* on X is $\mathcal{Y} \subseteq 2^X$ s.t. $\mathcal{Y}$ is closed for complements, countable unions and countable intersections, i.e.:
  - $\forall Y \in \mathcal{Y}.\ \bar{Y} \in \mathcal{Y}$, with $\bar{Y} = \{x \in X \mid x \notin Y\}$ being the complement of Y
  - $\forall I \subseteq \mathbb{N}$ s.t. $Y_i \in \mathcal{Y}$:
    - $\forall i, j \in I.\ Y_i \cup Y_j \in \mathcal{Y}$
    - $\forall i, j \in I.\ Y_i \cap Y_j \in \mathcal{Y}$
- Example: $\mathcal{Y}_1 = 2^X$ and $\mathcal{Y}_2 = \{\varnothing, X\}$ are always σ-algebras
- Example: for $X = \{a, b\}$, $\mathcal{Y} = \{\{a\}, \{b\}, \{a, b\}\}$ is not a σ-algebra since $\{a\} \cap \{b\} = \varnothing \notin \mathcal{Y}$

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Some Background

- If $X \subseteq \mathbb{R}^n$, the *Borel set* on X, denoted by $\mathcal{B}(X)$, is the smallest σ-algebra of X which contains all open sets of X
  - recall that a set $A \subseteq \mathbb{R}^n$ is open iff, for all $a \in A$, there exists a n-dimensional ball (border excluded) centered in a which is contained in A
  - that is, $\exists \varepsilon > 0 : \ \forall x \in \mathbb{R}^n. \, |a - x| < \varepsilon \Rightarrow x \in A$
- The pair $(X, \mathcal{B}(X))$ is called *measureable space*
- Thus, given $X \subseteq \mathbb{R}^n$, $\mathcal{B}(X)$ retains all open sets already in X and ensures that intersection, union and complementation are still in $\mathcal{B}(X)$
- We are interested in this since our systems are defined via variables on real intervals
  - sets of states are subsets of $\mathbb{R}^n$

Some Background

- A *stochastic kernel* on $(X, \mathcal{B}(X))$ is a function $K : X \times \mathcal{B}(X) \to [0, 1]$ s.t.:
  - for all $x \in X$, the function $K_X : \mathcal{B}(X) \to [0, 1]$ defined by $K_X(B) = K(x, B)$ is a probability measure on $\mathcal{B}$
    - that is, the three Kolmogorov axioms are true
    - note that K_X actually takes subsets...
  - for all $B \in \mathcal{B}(X)$, the function $K_{\mathcal{B}} : X \to [0, 1]$ is a measureable function on X
    - we are less interested on this point

Some Background

- Since each state is a point $x \in X \subseteq \mathbb{R}$, execution traces are sequences $\sigma \in X^\omega$
  - for finite (terminated) runs, we may add a loop on the last state (*stuttering*)
- We want to define probabilities on traces, thus $\Omega = X^\omega$
- Usually, we define the probability on $(\Omega, 2^\Omega)$
- For these types of Ω, we are happy with something contained in 2^Ω, namely $\mathcal{F}$ as the cylindric σ-algebra built on Ω
  - essentially, such sequences behave "well"

Some Background

- We suppose to have a stochastic kernel K defined on $(\Omega, \mathcal{F})$
- Together with an initial state $x \in X$, this defines a probability on $(\Omega, \mathcal{F})$
    - $\mathbb{P}(X_1 \in B) = 1$ if $x \in B$ and 0 otherwise;
    - $\mathbb{P}(X_{i+1} \in B) = K(x_i, B)$
        - K defines the outgoing transitions probability
- Thus, if we are able to define a K, we have a probability space for our SMC methodology

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Discrete Time Hybrid Automaton

- Giving a precise semantics of Simulink (or Modelica) is difficult, but the following definition is quite close
- A *Discrete Time Hybrid Automaton* (DTHA) is defined as $\mathcal{D} = \langle Q, E, n, q_0, x_0, \Phi, J \rangle$ where:
  - n is the dimension of the state space, which is understood to be $\mathbb{R}^n$
  - (Q, E) is a directed graph
    - Q is a set of *locations*, E is a set of *control switches* or *modalities*
  - (q_0, x_0) is the *starting state*, $(q_0, x_0) \in Q \times \mathbb{R}^n$
  - $\Phi = \{\phi_q : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n \mid q \in Q\}$
  - $J = \{j_e : \mathbb{R}^n \to \mathbb{R}^n \mid e \in E\}$

Discrete Time Hybrid Automaton Semantics

- The *transition relation* δ of a DTHA $\mathcal{D}$ defines when we go from a state in $Q \times \mathbb{R}^n$ to another
    - not simple as for Kripke structures, where one step is one step: here, also time passing is important
- 2 underling ideas:
    - time only passes within locations, handled by Φ
    - jumps within locations happen in time 0, defined by E with conditions given by J
    - either the time pass within a location, or a jump between locations is performed
- $\delta \in Q \times \mathbb{R}^n \times (\mathbb{R}^+ \dot\cup E) \times Q \times \mathbb{R}^n$

Discrete Time Hybrid Automaton Semantics

- $\delta \in Q \times \mathbb{R}^n \times (\mathbb{R}^+ \dot\cup E) \times Q \times \mathbb{R}^n$
- $(q, x, t, q, x') \in \delta \equiv (q, x) \rightarrow_t (q, x')$ iff $x' = \phi_q(t, x)$
  - note that q does not change
- $(q, x, e, q', x') \in \delta \equiv (q, x) \rightarrow_e (q', x')$ iff $x' = j_e(x) \wedge e = (q, q')$
  - note that time does not pass
- $\Delta : Q \times \mathbb{R}^n \rightarrow (\mathbb{R}^+ \dot\cup E)$ is the *simulation function*
  - decides if, in a given state, a location jump or a time pass has to be performed
  - if time passes, decides how much
  - unified notation $(q, x) \rightarrow_{\Delta(q,x)} (q', x')$

Discrete Time Hybrid Automaton Semantics

- δ may be non-deterministic
  - in a given state (q, x), both some j_e and ϕ_q could be enabled
  - even if only ϕ_q is enabled, many values for t may apply
- Δ is deterministic; in Simulink:
  - if both a discrete and a continuous transition can be taken, take the discrete one
  - if continuous, stay for the maximum time allowed before a location change
  - an ordering on outgoing edge is always available, so the first one is selected when multiple edges are present

Discrete Time Hybrid Automaton Semantics

- A *trace* is a sequence $\sigma = (s_0, t_0), \ldots, (s_i, t_i), \ldots$ s.t.
    - $s_0 = (q_0, x_0)$
    - $\forall i \geq 0.\ s_i \in Q \times \mathbb{R}^n,\ t_i \in \mathbb{R}^+$
    - $\forall i \geq 0.\ s_i \rightarrow_{\Delta(s_i)} s_{i+1}$
    - $\forall i \geq 0.\ t_i = \Delta(s_i)$ if $\Delta(s_i) \in \mathbb{R}^+$
    - $\forall i \geq 0.\ t_i = 0$ if $\Delta(s_i) \in E$
- At step $\sigma_i = (s_i, t_i)$, the global time is $\sum_{j=0}^{i-1} t_j$
- For an infinite trace σ, $\sum_{j=0}^{\infty} t_j = \infty$
    - there must be finitely many location switches in finite time

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Probabilistic Discrete Time Hybrid Automaton Semantics

- For a set X, let $D(X) = \{f \mid f$ is a probability density function on $X\}$
  - for $X = \{x_1, \ldots, x_n\}$, $f(t) = \sum_{i=1}^{n} p_i \delta(t - x_i)$, for any choice of $\pi_i \in [0, 1]$ s.t. $\sum_{i=1}^{n} p_i = 1$
  - here δ is the Dirac function, i.e. $\delta(0) = 1, \forall x \neq 0.\ \delta(x) = 0$
  - otherwise, for continuous X, $f(t)$ is s.t. $\int_a^b f(x)dx \in [0, 1]$ for any $[a, b] \subseteq X$ and $\int_X f(x)dx = 1$
- A *probabilistic transition function* Π for a DTHA $\mathcal{D}$ is a function $\Pi : Q \times \mathbb{R}^n \to D(\{0, 1\}) \times D(\mathbb{R}^+) \times D(E)$
  - since Π returns 3 values, we will denote its components by $\Pi(s) = \langle \Pi_a(s), \Pi_c(s), \Pi_d(s) \rangle$
  - the following must be true: $\forall (q, q') \in E, r \in Q, x \in \mathbb{R}^n.\ q \neq r \to \Pi_d(r, x)(q, q') = 0$

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Probabilistic Discrete Time Hybrid Automaton Semantics

- Informally, a probabilistic transition function Π has the goal of definining a (possibly non-uniform) “random walk” on a DTHA
  - suppose we are in a state $s = (q, x)$
  - both a location change and a continuous move may be taken? choose at random with probability $\Pi_a(s)$
  - if a location change must take place, choose one at random with probability $\Pi_d(s)$
  - if time must pass, decide how much with probability $\pi_c(s)$

Probabilistic Discrete Time Hybrid Automaton Semantics

- Thus, $K((q,x),B) = p_a \sum_{e \in E(B,q,x)} \Pi_d(q,x)(e) + (1 - p_a) \int_0^\infty \Pi_c(q,x)(t) I_B(q, \phi_q(t,x))dt$
  - B is a Borel set over $Q \times \mathbb{R}^n$
  - $p_a = \Pi_a(q,x)(0)$
    - arbitrary choice, could also have been $p_a = \Pi_a(q,x)(1)$
  - $E \supseteq E(B,q,x) = \{(q,q') \in E \mid (q', j_{(q,q')}(x)) \in B\}$
    - to be well-defined, we must stay in the same Borel set
  - I_B is the indicator function of B, i.e. $I_B(q,x) = 1$ iff $(q,x) \in B$, and 0 otherwise
    - again, to be well-defined, we must stay in the same Borel set
- It may be shown that K is a stochastic kernel, so probability is well-defined over infinite traces

Probabilistic DTHA in Simulink/Stateflow

- n is the number of variables in a Simulink/Stateflow model
    - some of them may be discrete, but $\mathbb{R}^n$ is for sure a superclass
- Q corresponds to "states" of Stateflow and E are states transitions
- Simulink only perform deterministic transitions, so probability density function output by Π all consists in just one point being defined
- Differently from the Grosu & Smolka works, here we cannot provide a deterministic model and let the methodology turn it probabilistic
    - the user must define something probabilistic
    - typically done by introducing probabilistic blocks in the design
    - Uniform Random Number block

Bayes Theorem

- Conditional probability: $\mathbb{P}(A|B)$ is the probability of event A, under the assumption that event B already occured
  - by definition, $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$
- Which is the relationship between $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$?
- The well-known Bayes Theorem states that $\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$
  - $\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)}$ for $\cup_{i=1}^{n} A_i = B$
- Here we need a more refined version of the Bayes Theorem
- First of all, the conditional probability density function of a Bernoulli random variable X and a random variable U with values in $(0,1)$ is $f(x_i|u) = u^{x_i}(1-u)^{1-x_i}$
  - then, $f(x_i = 1|u) = u$ and $f(x_i = 0|u) = 1 - u$

Bayes Theorem

- Our refined version of the Bayes Theorem states that $f(u|x_1 \dots x_n) = \frac{f(x_1 \dots x_n|u)g(u)}{\int_0^1 f(x_1 \dots x_n|v)g(v)dv}$
  - u is the unknown probability that we have an error in our system
  - x_i are "observations" of u: we make a simulation and see if it fails or not
  - g is the probability *prior* distribution of u
    - prior as opposed to *posterior* $f(u|x_1 \dots x_n)$: without having taken samples
    - we will assume it to have a given shape
  - since we assume observations to be independent, $f(x_1 \dots x_n|u) = \prod_{i=1}^{n} f(x_i|u)$
- We want to know p as the probability of the posterior $f(u|x_1 \dots x_n)$
- We use the *posterior Bayes estimator* of p

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Bayes Theorem

- From the Bayes theorem it follows that $\int_{t_0}^{t_1} f(u|x_1 \dots x_n)du = F_{(x+\alpha,n-x+\beta)}(t_1) - F_{(x+\alpha,n-x+\beta)}(t_0)$ where:
  - $t_0, t_1 \in (0,1)$
  - $x = \sum_{i=1}^{n} x_i$ is the number of successes in the n trials
  - $\alpha, \beta \in \mathbb{R}^+$ are given parameters
  - $F_{A,B}(t) = \int_0^t g_{A,B}(u)du$ is a Beta distribution function
  - g above is the prior density, here we assume it to be $g_{A,B}(u) = \frac{u^{A-1}(1-u)^{B-1}}{\int_0^1 t^{A-1}(1-t)^{B-1}dt}$
  - thus, $F_{A,B}(t) = \frac{\int_0^t u^{A-1}(1-u)^{B-1}du}{\int_0^1 t^{A-1}(1-t)^{B-1}dt}$
  - F may be easily made explicit, or simply computed using mathematical tools like MATLAB
- When sampling from a Bernoulli distribution with a Beta prior of parameters α, β, it is known that the mean of the posterior is $\hat{p} = \frac{x+\alpha}{n+\alpha+\beta}$

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

The Algorithm for BSMC

- BSMC: Bayes-based Statistical Model Checking
- The input is as follows:
  - $\mathcal{S}$ as the simulator model for the system to be verified
    - may be black-box, Simulink, Modelica or proprietary
    - must have some probabilistic behaviour, i.e., 2 consecutive simulations may have different results
  - φ as the BLTL property to be verified
    - Bounded LTL: all **U** operators must be bounded, i.e., they are of the form $\mathbf{U}^{\leq t}$, with $t > 0$
    - hence, also **F** and **G** must be bounded too
  - $\alpha, \beta \in \mathbb{R}^+$ as the parameters for the prior Beta distribution
  - $\delta \in (0, 1)$ as the desired size of the output interval
  - $c \in (\frac{1}{2}, 1)$ as the desired interval coverage coefficient

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

The Algorithm for BSMC

- The output is as follows:
  - (t_0, t_1) such that $t_1 - t_0 = \delta$
  - $\hat{p}$ as the estimate of the probability p that $\mathcal{S} \models \varphi$
- It holds that:
  - (t_0, t_1) is a $100c$ *Bayesian interval estimate*
  - $\hat{p} \in (t_0, t_1)$
    - usually at half interval, but with some adjustments
- Thus, we want δ to be small
  - implies our output interval is narrow, and the estimate is accurate
- We want c to be high
  - implies we are confident on the estimate
- Needless to say, the smaller δ and the higher c, the higher computation time required

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

The Algorithm

```
BSMC(ProbModel $\mathcal{S}$, BLTL property $\varphi$,
     double $\alpha, \beta, c, \delta$) {
  $(n, x)$ = (0, 0);
  do {
    $\sigma$ = simulate($\mathcal{S}$, time($\varphi$));
    $n$ = $n+1$; if ($\sigma \models \varphi$) $x$ = $x+1$;
    $\hat{p}$ = $\frac{x+\alpha}{n+\alpha+\beta}$; $(t_0, t_1)$ = $(\hat{p} - \frac{\delta}{2}, \hat{p} + \frac{\delta}{2})$;
    if ($t_1$ > 1) $(t_0, t_1)$ = $(1 - \delta, 1)$;
    if ($t_0$ < 0) $(t_0, t_1)$ = $(0, \delta)$;
    $\gamma$ = $F_{(x+\alpha, n-x+\beta)}(t_1) - F_{(x+\alpha, n-x+\beta)}(t_0)$;
  } while ($\gamma$ < $c$);
  return $\langle (t_0, t_1), \hat{p} \rangle$;
}
```

BLTL Logic, Formally

$$\Phi ::= p \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid (\Phi) \mid \Phi_1 \ \mathbf{U}^{\leq t}\ \Phi_2$$

- $t \in \mathbb{Q}^+$ is a time
- Atomic propositions p are of the form $y \sim v$, being y a variable in the model, $\sim \in \{<, >, \leq, \geq, =\}$ and $v \in \mathbb{Q}$
- Some other derived operators:
  - of course true, false, OR and other propositional logic connectors
  - future (or eventually): $\mathbf{F}^{\leq t}\Phi = \mathrm{true}\ \mathbf{U}^{\leq t}\ \Phi$
  - globally: $\mathbf{G}^{\leq t}\Phi = \neg(\mathrm{true}\ \mathbf{U}^{\leq t}\ \neg\Phi)$
- As for LTL, $\mathcal{S} \models \varphi$ when, for all executions σ of $\mathcal{S}$, σ satisfies φ
- For a given σ, $\sigma \models \varphi$ iff $\sigma, 0 \models \varphi$

BLTL Logic, Formally

- To define when $\sigma, i \models \varphi$, a recursive definition over the recursive syntax of BLTL is provided
  - recall that $\sigma = (s_0, t_0), \ldots, (s_i, t_i), \ldots$
  - at step $\sigma_i = (s_i, t_i)$, the global time is $\sum_{j=0}^{i-1} t_j$
- $\sigma, i \models y \sim v$ iff $\sigma(i)(y) \sim v$
- $\sigma, i \models \Phi_1 \wedge \Phi_2$ iff $\sigma, i \models \Phi_1 \wedge \sigma, i \models \Phi_2$
- $\sigma, i \models \neg\Phi$ iff $\sigma, i \not\models \Phi$
- $\sigma, i \models \Phi_1 \mathbf{U}^{\leq t} \Phi_2$ iff
  $\exists k \geq i : \ \sigma, k \models \Phi_2 \wedge \forall i \leq j < k.\ \sigma, j \models \Phi_1$ and $\sum_{j=i}^{k-1} t_j \leq t$
- Note this is different from the bounded semantics of LTL used in Bounded Model Checking

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

On BSMC Algorithm

- Crucial steps in BSMC algorithm:
  - `simulate`, i.e., invoking our simulator, whatever it is
  - evaluating $\sigma \models \varphi$
- Does `simulate` actually returns σ?
  - typically, simulators output is a log with lines $(t_i, v_{i1}, \ldots, v_{in})$
  - being $v_{i1}, \ldots, v_{in}$ the values at time t_i for each of the n variables used in the simulator model
  - usually, state locations may be inferred from $v_{i1}, \ldots, v_{in}$
  - usually, $t_{i+1} = t_i + \Delta t$ for a fixed (and small) $\Delta t > 0$
  - thus, a simple postprocess computation may translate the log in an execution $\sigma = (s_0, t_0), \ldots, (s_i, t_i), \ldots$
  - this also allows to compute $\sigma(i)(y)$ for any variable y

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

On BSMC Algorithm

- The first 2 inputs of the BSMC algorithm are straightforward
  - if I want to verify something, of course I need a model and a property
- We may understand δ, c: they control accuracy and confidence of the result
  - the more accuracy/confidence is required, the longer the computation
- What about α, β?
  - informally, it is a measure of the "weight" we *believe* passes and fails should have
  - if none is known, it is probably good to choose a *uniform* Beta distribution, i.e., $\alpha = \beta$
  - e.g., $\alpha = \beta = 1$

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

On BSMC Algorithm

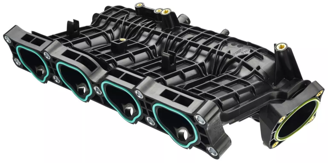
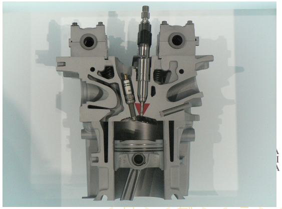
- It may be proven that BSMC nearly always terminates:
  - for all possible valid inputs, BSMC terminates with probability 1
  - no, this does not imply that BSMC always terminates (we are in an infinite space)
  - but it is enough for practical applications
- It may be proven that errors on BSMC output are unlikely
  - let our null hypothesis be $p \in (t_0, t_1)$
  - both type-I and type-II errors are bounded by $\frac{\pi_0(1-c)}{c(1-\pi_0)}$
    - recall: type-I is saying that $p \notin (t_0, t_1)$ when instead $p \in (t_0, t_1)$
    - recall: type-II is saying that $p \in (t_0, t_1)$ when instead $p \notin (t_0, t_1)$
  - c is the coverage input as in BSMC
  - π_0 is the actual (prior) probability that $p \in (t_0, t_1)$

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

BSMC Results

- Case study: Fault-Tolerant Fuel Control System
  - for details, see `http://www.mathworks.com/help/simulink/examples/modeling-a-fault-tolerant-fuel-control-system.html`
- Gasoline engine (e.g., used in avionics), must provide power for vehicle operations
- This model focuses on a critical parameter: the air/fuel rate, which must be kept close to a reference value, i.e., 14.6
  - air is pumped away by intake manifold, fule is pumped in by injectors

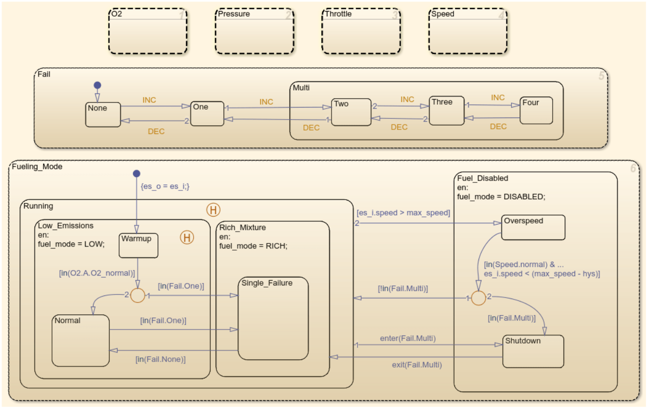
BSMC Results

- The model uses sensors for some key measurements: EGO (exhaust gas residual oxygen), engine speed, throttle, pressure
- If all sensors works well, it is rather easy to control the actuators so that the air/fuel ratio is 14.6
  - the actuator is on the fuel rate
- But sensors may fail: the controller is able to detect such failures and adjust actuators accordingly
- If more than one sensor fail, the engine is shut down
- We need a stochastic system, thus sensor failures are made probabilistic
  - independent Poisson processes with different arrival rates: $\mathbb{P}(N(t) = n) = \frac{\lambda^n t^n}{n! e^{\lambda t}}$
- The other parts of the system are deterministic
  - there should be the throttle command as input, but it is replaced by a triangular deterministic input

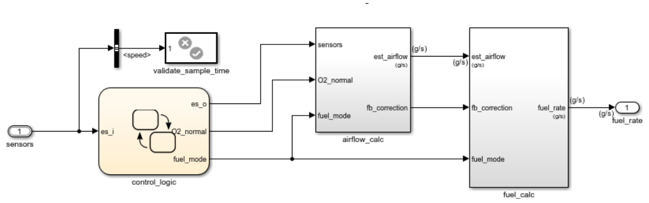
UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

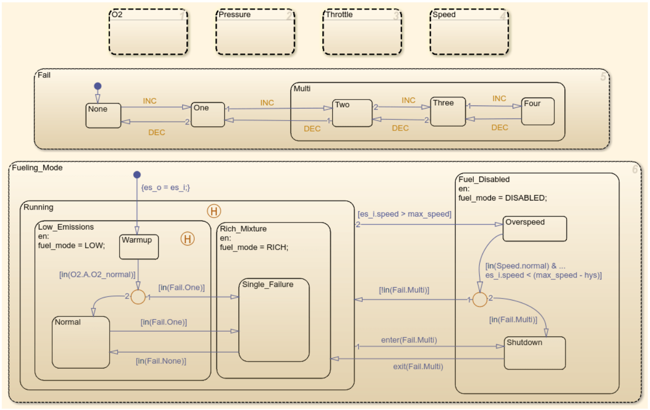
BSMC Results



BSMC Results



BSMC Results



BSMC Results

- BLTL formula to be checked: $\neg \mathbf{F}^{100}\mathbf{G}^{1}\mathrm{FuelFlowRate} = 0$
  - it must not happen that, within 100 seconds, the fuel flow rate becomes zero for 1 second
- Different experiments varying:
  - $(\delta, c) \in \{0.05, 0.01\} \times \{0.99, 0.999\}$ (4 possible pairs)
  - fault rates for sensors in $\{(3, 7, 8), (10, 8, 9), (20, 10, 20), (30, 30, 30)\}$
- The C-H bound is also computed: how many experiments should be done with the Chernoff-Hoeffding methodology

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DISIM Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

BSMC Results

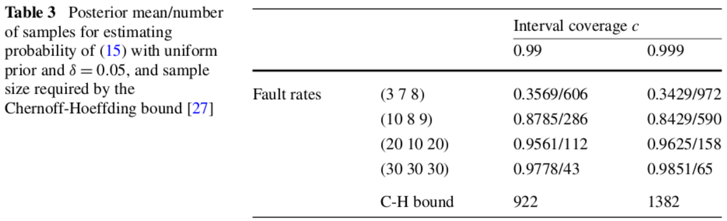


Table 3 Posterior mean/number of samples for estimating probability of (15) with uniform prior and $\delta = 0.05$, and sample size required by the Chernoff-Hoeffding bound [27]

		Interval coverage c	
		0.99	0.999
Fault rates	(3 7 8)	0.3569/606	0.3429/972
	(10 8 9)	0.8785/286	0.8429/590
	(20 10 20)	0.9561/112	0.9625/158
	(30 30 30)	0.9778/43	0.9851/65
	C-H bound	922	1382

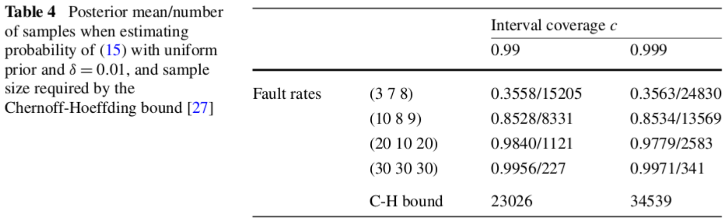


Table 4 Posterior mean/number of samples when estimating probability of (15) with uniform prior and $\delta = 0.01$, and sample size required by the Chernoff-Hoeffding bound [27]

		Interval coverage c	
		0.99	0.999
Fault rates	(3 7 8)	0.3558/15205	0.3563/24830
	(10 8 9)	0.8528/8331	0.8534/13569
	(20 10 20)	0.9840/1121	0.9779/2583
	(30 30 30)	0.9956/227	0.9971/341
	C-H bound	23026	34539